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LAYERED COMPOSITES WITH AN INTERFACE FLAW†

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Abstract—The plane strain problem for a bonded medium composed of three different materials is considered. It is assumed that the medium contains a flaw on one of the interfaces which may be idealized as a crack. The integral equations for the general problem are obtained, which turn out to be a system of singular integral equations of the second kind. The singularity of the system is removed and the equations are solved by taking advantage of the fact that the fundamental function of the integral equations is the weight function of Jacobi polynomials. The problems for two half-planes joined through a layer, an elastic layer bonded to a half-plane and two bonded layers are solved as specific examples, and numerical results involving the stress intensity factors and the strain energy release rate are presented.

1. INTRODUCTION

THE general plane elasticity problem for a multi-layered composite medium containing a crack was considered in a previous paper [1]. Even though the general procedure for deriving the integral equations of the problem was outlined in [1], the main effort in that paper was devoted to the analysis of the stress disturbance resulting from a crack which is imbedded in a homogeneous elastic layer bonding two elastic half-planes. In that case the problem was reduced to the solution of a system of singular integral equations of the first kind. Since the nature of stress singularities for a crack imbedded in a homogeneous medium and for an interface crack is different, there is no smooth transition from one solution to the other as the crack distance from the interface goes to zero. For very small values of this distance one also encounters convergence problems in the numerical analysis. This is primarily due to the fact that as the crack-to-interface distance goes to zero, the Fredholm kernels in the system become unbounded. For the interface crack if one separates these singular parts of the kernels, the system of singular integral equations become one of the second kind the solution of which requires a different numerical technique.

Physically, it is obvious that any manufacturing flaw that exists would be either in the bonding layer or, perhaps more likely, on the interface. Thus, to complete the analysis of bonded layers with a flaw, it is necessary to have the solution of the interface crack problem.

In this paper we will consider the plane strain (or the generalized plane stress) problem for the bonded medium which is composed of three different materials and which contains an interface crack (Fig. 1). Specifically we will be concerned with the effect of the ratio of the layer thickness to the crack length on the stress intensity factors and the strain energy release rate, the latter being the main "load parameter" in the application of the fracture criterion. The particular examples which will be considered are two half-planes joined through a layer ($h_2 = \infty$ in Fig. 1), an elastic layer bonded to a half-plane ($h_2 = \infty, \mu_1 = 0$), and two bonded layers (h, h_2 finite, $\mu_1 = 0$).

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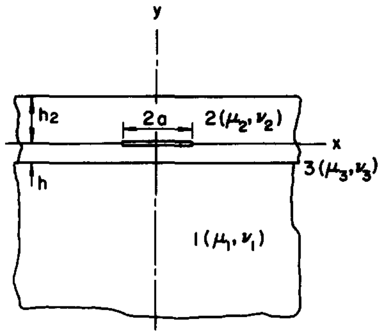

FIG. 1. The geometry and notation.

2. DERIVATION OF THE INTEGRAL EQUATIONS

Consider the plane problem for the layered medium shown in Fig. 1. Let the medium contain a crack on one of the interfaces. Without any loss in generality, the half-length of the crack is assumed to be unity. In this paper we are primarily interested in the disturbed stress state caused by the crack. Thus, assuming that the overall stress distribution σ_{ij}^0, in the imperfection-free medium is known, the stress state σ_{ij}^t in the cracked medium may be expressed as

$$\sigma_{ij}^t = \sigma_{ij}^0 + \sigma_{ij} \tag{1}$$

where σ_{ij} is the disturbed state which may be obtained by using the tractions (Fig. 1)

$$p_1(x) = -\sigma_{yy}^0(x, 0), \qquad p_2(x) = -\sigma_{xy}^0(x, 0), \qquad |x| < 1 \tag{2}$$

as the only external loads. Because of the symmetry with respect to the $x = 0$ plane, the general problem can always be expressed as the sum of a symmetric component and an antisymmetric component. Here, we will further restrict our attention to the symmetric problem for which the tractions p_i have the following properties

$$p_1(x) = p_1(-x), \qquad p_2(x) = -p_2(-x), \qquad |x| < 1. \tag{3}$$

The treatment of the anti-symmetric problem requires only a slight modification.

Let u_i, v_i be the x, y-components of the displacement vector in the ith material shown in Fig. 1. Expressing u_i, v_i as appropriate Fourier integrals, as shown in [1], the problem may be formulated in terms of two unknown functions defined by

$$f_1(x) = \frac{\partial}{\partial x}(u_2^+ - u_3^-), \qquad f_2(x) = \frac{\partial}{\partial x}(v_2^+ - v_3^-) \tag{4}$$

where the superscripts + and − refer to the limiting values of the displacements as y approaches zero from + and − sides, respectively.

Thus, referring to [1] for details, the components of the stress vector at $y = 0$ and $x > 0$ may be expressed as

$$-\frac{1+\kappa_3}{\mu_3}\sigma_{yy}^3(x,0) = \lim_{y\to 0^-}\frac{2}{\pi}\int_0^\infty \sum_1^2 a_{1j}e^{\alpha y}A_j(\alpha)\cos\alpha x\,d\alpha + \frac{2}{\pi}\int_0^\infty \sum_1^2 H_{1j}(\alpha)A_j(\alpha)\cos\alpha x\,d\alpha$$

$$-\frac{1+\kappa_3}{\mu_3}\sigma_{xy}^3(x,0) = \lim_{y\to 0^-}\frac{2}{\pi}\int_0^\infty \sum_1^2 a_{2j}e^{\alpha y}A_j(\alpha)\sin\alpha x\,d\alpha + \frac{2}{\pi}\int_0^\infty \sum_1^2 H_{2j}(\alpha)A_j(\alpha)\sin\alpha x\,d\alpha \tag{5}$$

where A_j are the Fourier transforms of f_j defined as follows:

$$A_1(\alpha) = \int_0^\infty f_1(t)\cos\alpha t\,dt, \qquad A_2(\alpha) = \int_0^\infty f_2(t)\sin\alpha t\,dt. \tag{6}$$

The constants a_{ij} depend on the elastic properties of the materials adjacent to the crack only and are given by

$$a_{11} = -a_{22} = (1+\lambda_2\lambda_4)/\lambda_4, \qquad a_{12} = -a_{21} = -(1+2\lambda_4-\lambda_2\lambda_4)/\lambda_4$$
$$\lambda_2 = (\kappa_2\mu_3-\kappa_3\mu_2)/(\mu_2+\kappa_2\mu_3) \tag{7}$$
$$\lambda_4 = (\mu_3+\mu_2\kappa_3)/(\mu_2-\mu_3)$$

where μ_i is the shear modulus and $\kappa_i = 3-4\nu_i$ for plane strain and $\kappa_i = (3-\nu_i)/(1+\nu_i)$ for generalized plane stress, ν_i being the Poisson's ratio. For the different geometries considered in this paper the functions $H_{ij}(\alpha)$ are given in the Appendix.

Note that in (5) $y < 0$ and for $\alpha \to \infty$ $H_{ij} \sim 0(e^{-2\alpha h})$. Thus the integrals on the right-hand side are uniformly convergent; as a result, certain operations such as change of order of integration are permissible. Also note that once the dislocations $f_i(x)$ on the interface are specified, (5) with (6) gives the stresses for all values of x. In the crack problem under consideration $f_i(x)$ are zero for $|x| > 1$ and are unknown for $|x| < 1$. On the other hand the stress vector on the interface $y = 0$ is unknown for $|x| > 1$ and is given by the following known functions for $|x| < 1$:

$$\sigma_{yy}^3(x,0) = p_1(x), \qquad \sigma_{xy}^3(x,0) = p_2(x), \qquad |x| < 1. \tag{8}$$

Using these informations, substituting from (6) into (5), and also using the expressions of the following form resulting from the symmetry properties $f_1(t) = f_1(-t)$, $f_2(t) = -f_2(-t)$,

$$\int_0^\infty H(\alpha)\cos\alpha x\,d\alpha\int_0^1 f_1(t)\cos\alpha t\,dt = \frac{1}{2}\int_{-1}^1 f_1(t)\,dt\int_0^\infty H(\alpha)\cos\alpha(t-x)\,d\alpha$$

we obtain

$$
\begin{aligned}
-\frac{1+\kappa_3}{\mu_3}p_1(x) = \lim_{y\to 0^-}\Bigg[&\frac{a_{11}}{\pi}\int_{-1}^{1} f_1(t)\,\mathrm{d}t\int_0^\infty \mathrm{e}^{\alpha y}\cos\alpha(t-x)\,\mathrm{d}\alpha \\
&+\frac{a_{12}}{\pi}\int_{-1}^{1} f_2(t)\,\mathrm{d}t\int_0^\infty \mathrm{e}^{\alpha y}\sin\alpha(t-x)\,\mathrm{d}\alpha\Bigg] \\
&+\frac{1}{\pi}\int_{-1}^{1}\sum_1^2 k_{1j}(x,t)f_j(t)\,\mathrm{d}t \\
-\frac{1+\kappa_3}{\mu_3}p_2(x) = \lim_{y\to 0^-}\Bigg[&-\frac{a_{21}}{\pi}\int_{-1}^{1} f_1(t)\,\mathrm{d}t\int_0^\infty \mathrm{e}^{\alpha y}\sin\alpha(t-x)\,\mathrm{d}\alpha \\
&+\frac{a_{22}}{\pi}\int_{-1}^{1} f_2(t)\,\mathrm{d}t\int_0^\infty \mathrm{e}^{\alpha y}\cos\alpha(t-x)\,\mathrm{d}\alpha\Bigg] \\
&+\frac{1}{\pi}\int_{-1}^{1}\sum_1^2 k_{2j}(x,t)f_j(t)\,\mathrm{d}t, \qquad |x|<1
\end{aligned}
\tag{9}
$$

where the bounded kernels k_{ij} are given by

$$
\begin{aligned}
k_{11}(x,t) &= \int_0^\infty H_{11}(\alpha)\cos\alpha(t-x)\,\mathrm{d}\alpha \\
k_{12}(x,t) &= \int_0^\infty H_{12}(\alpha)\sin\alpha(t-x)\,\mathrm{d}\alpha \\
k_{21}(x,t) &= \int_0^\infty H_{21}(\alpha)\sin\alpha(t-x)\,\mathrm{d}\alpha \\
k_{22}(x,t) &= \int_0^\infty H_{22}(\alpha)\cos\alpha(t-x)\,\mathrm{d}\alpha.
\end{aligned}
\tag{10}
$$

Evaluating the infinite integrals in (9), passing to limit [2] and dividing by $-a_{21}$ we finally obtain

$$
\begin{aligned}
\frac{1+\kappa_3}{a_{21}\mu_3}p_1(x) &= \gamma f_1(x)+\frac{1}{\pi}\int_{-1}^{1}\frac{f_2(t)}{t-x}\,\mathrm{d}t \\
&\quad -\frac{1}{a_{21}}\frac{1}{\pi}\int_{-1}^{1}\sum_1^2 k_{1j}(x,t)f_j(t)\,\mathrm{d}t \\
\frac{1+\kappa_3}{a_{21}\mu_3}p_2(x) &= \frac{1}{\pi}\int_{-1}^{1}\frac{f_1(t)\,\mathrm{d}t}{t-x}-\gamma f_2(x) \\
&\quad -\frac{1}{a_{21}}\frac{1}{\pi}\int_{-1}^{1}\sum_1^2 k_{2j}(x,t)f_j(t)\,\mathrm{d}t, \qquad -1<x<1
\end{aligned}
\tag{11}
$$

where

$$
\gamma = \frac{a_{11}}{a_{12}} = \frac{(\mu_3+\kappa_3\mu_2)-(\mu_2+\kappa_2\mu_3)}{(\mu_2+\kappa_2\mu_3)+(\mu_3+\kappa_3\mu_2)}
\tag{12}
$$

(11) provides the system of integral equations to determine the unknown functions f_i. Once these functions are obtained all the desired field quantities in the medium can be expressed as and evaluated from definite integrals involving $f_i(t)$ and the appropriate kernels. Note that the continuity of displacements along the bonded portion of the interface, i.e.

$$u_2^+ - u_3^- = 0, \qquad v_2^+ - v_3^- = 0, \qquad |x| > 1, \qquad y = 0$$

requires that, in addition to $f_1 = 0 = f_2$ for $|x| > 1$, which is used to derive the integral equations, f_1 and f_2 must satisfy the following conditions:

$$\int_{-1}^{1} f_1(x)\,\mathrm{d}x = 0, \qquad \int_{-1}^{1} f_2(x)\,\mathrm{d}x = 0. \tag{13}$$

From the expressions given in the Appendix it is seen that as h and h_2 go to infinity H_{ij}, and as a result, k_{ij} go to zero and (11) reduces to its dominant system representing two bonded half-planes with an interface crack [2]. In addition to $h = \infty = h_2$ if we also let $\mu_2 = \mu_3$, $\kappa_2 = \kappa_3$ we have $\gamma = 0$, $a_{21} = 2$ and we recover the simple (uncoupled) singular integral equations for a homogeneous infinite plane with a crack [1].

3. THE SOLUTION OF THE INTEGRAL EQUATIONS

The system of singular integral equations similar to (11) has been extensively studied in [3]. To simplify the solution we combine the two equations given by (11) as follows:

$$\frac{1}{\pi i}\int_{-1}^{1} \frac{\phi(t)\,\mathrm{d}t}{t-x} - \gamma\phi(x) + \int_{-1}^{1} [K_1(x,t)\phi(t) + K_2(x,t)\overline{\phi(t)}]\,\mathrm{d}t = g(x), \qquad -1 < x < 1 \tag{14}$$

where

$$\begin{aligned} \phi(x) &= f_2(x) + if_1(x) \\ K_1(x,t) &= \frac{1}{2\pi a_{21}}[(k_{11}-k_{22}) + i(k_{12}+k_{21})] \\ K_2(x,t) &= \frac{1}{2\pi a_{21}}[-(k_{11}+k_{22}) + i(k_{12}-k_{21})] \\ g(x) &= \frac{1+\kappa_3}{a_{21}\mu_3}(p_2 - ip_1). \end{aligned} \tag{15}$$

The kernels K_1 and K_2 are bounded. Hence, aside from a multiplicative constant, the singular behavior of the function $\phi(t)$ at ∓ 1 is determined by the dominant part of the singular integral equation. The integral equation (14) will be solved under the assumption that ϕ satisfies a Hölder condition on every closed part of the interval $(-1, 1)$ not containing the ends, and its behavior near the ends is such that it may be represented by

$$\phi(x) = w(x)\psi(x), \qquad w(x) = (1-x)^{\alpha}(1+x)^{\beta}, \qquad |x| < 1 \tag{16}$$

where the function ψ is Hölder-continuous in the closed interval $[-1, 1]$ and†

$$-1 < \mathrm{Re}\,\alpha < 0, \qquad -1 < \mathrm{Re}\,\beta < 0. \tag{17}$$

† Physically this means that the displacement derivatives $f_1(x)$ and $f_2(x)$ are continuous in the open interval $-1 < x < 1$ and have integrable singularities at $x = \mp 1$.

The fundamental function, $w(x)$, of the integral equation may be obtained, within a mutliplicative constant, from the homogeneous dominant part given by

$$\frac{1}{\pi i}\int_{-1}^{1}\frac{w(t)\,dt}{t-x}-\gamma w(x) = 0. \tag{18}$$

Ignoring the constant the solution of (18) satisfying (17) may be written as [3, 4]

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}$$

$$\alpha = -\tfrac{1}{2}-i\omega, \qquad \beta = -\tfrac{1}{2}+i\omega, \qquad \omega = \frac{1}{2\pi}\log\left(\frac{1+\gamma}{1-\gamma}\right). \tag{19}$$

To solve the singular integral equation (14), rather than following the regularization methods described in [3] or [5] which, in this case, become extremely cumbersome, we will follow the technique described in [4]. Noting that the fundamental function, $w(x)$ of (14) is the weight of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, we will express the solution in the following form†

$$\phi(x) = \sum_{0}^{\infty} c_n w(x) P_n^{(\alpha,\beta)}(x), \qquad |x| < 1. \tag{20}$$

Using the continuity conditions (13), i.e.

$$\int_{-1}^{1}\phi(x)\,dx = 0$$

and the orthogonality relations of the Jacobi polynomials [6, 7], i.e.

$$\int_{-1}^{1} w(x)P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)\,dx = \begin{cases} 0, n \neq m;\, n, m = 0, 1, 2, \ldots \\ \dfrac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\,\dfrac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\,\Gamma(n+\alpha+\beta+1)}, & n = m \end{cases} \tag{21}$$

and also observing that $P_0^{(\alpha,\beta)}(x) = 1$, it is easily seen that in (20) $c_0 = 0$.

Substituting now from (20) into (14) and using the relation [8]

$$\frac{1}{\pi i}\int_{-1}^{1} w(t)P_n^{(\alpha,\beta)}(t)\frac{dt}{t-x}-\gamma w(x)P_n^{(\alpha,\beta)}(x) = \frac{\sqrt{(1-\gamma^2)}}{2i}P_{n-1}^{(-\alpha,-\beta)}(x), \qquad |x| < 1 \tag{22}$$

the singularity of the integral equation is removed and we obtain

$$\sum_{1}^{\infty} c_n \frac{\sqrt{(1-\gamma^2)}}{2i}P_{n-1}^{(-\alpha,-\beta)}(x)+\int_{-1}^{1}\sum_{0}^{\infty}[c_nK_1(x,t)w(t)P_n^{(\alpha,\beta)}(t)+\bar{c}_nK_2(x,t)\overline{w(t)}\,\overline{P_n^{(\alpha,\beta)}(t)}]\,dt$$
$$= g(x) \qquad |x| \leq 1. \tag{23}$$

Equation (23) can be reduced to an infinite system of algebraic equations in the unknown complex constants c_n by using a weighted residual technique. In this case the appropriate weight functions are

$$(1-x)^{-\alpha}(1+x)^{-\beta}P_n^{(-\alpha,-\beta)}(x), \qquad n = 0, 1, 2, \ldots \tag{24}$$

† Which means that the continuous function $\psi(x)$ given in (16) may be expanded into a series of orthogonal polynomials $P_n^{(\alpha,\beta)}(x)$ in $|x| \leq 1$.

and the technique is equivalent to expanding both sides of (23) into infinite series in Jacobi polynomials and equating the corresponding coefficients. Due to the complicated nature of the kernels $K_i(x, t)$, in this problem it is not possible to study the regularity of the resulting infinite system. Thus, in solving the system of algebraic equations by the method of reduction, a numerical convergence procedure has to be incorporated into the analysis (see [4] and the first example of this paper).

In the special case of the two bonded half-planes with an interface crack, $H_{ij}(\alpha)$ and, as a result, the kernels $K_i(x, t)$ are zero. Then, for surface tractions

$$p_1(x) = -\sigma_0, p_2(x) = 0 \tag{25}$$

multiplying both sides of (23) by the functions (24) and integrating from -1 to $+1$ we find

$$c_1 = \frac{2i}{\sqrt{(1-\gamma^2)}}\frac{i\sigma_0(1+\kappa_3)}{a_{21}\mu_3}; \qquad c_n = 0, \qquad n = 2, 3, \ldots \tag{26}$$

which by (20) gives the exact solution as follows

$$\phi(x) = -\frac{2\sigma_0(1+\kappa_3)}{a_{21}\mu_3\sqrt{(1-\gamma^2)}}w(x)P_1^{(\alpha,\beta)}(x). \tag{27}$$

To obtain the contact stresses along the interface we observe that equations (5), (9), (11) and (14) give the stresses for $|x| > 1$ as well as $|x| < 1$. However, for $|x| > 1$ since

$$\lim_{y\to 0+}\int_{-1}^{1} f_j(t)\,\mathrm{d}t\int_0^\infty \mathrm{e}^{-\alpha y}\cos\alpha(t-x)\,\mathrm{d}\alpha = \lim_{y\to 0+}\int_{-1}^{1} f_j(t)\,\mathrm{d}t\frac{y}{y^2+(t-x)^2}$$
$$= \begin{cases}\pi f_j(x), & |x| < 1\\ 0, & |x| > 1\end{cases}; \qquad j = 1, 2 \tag{28}$$

from (14) and (27) we obtain

$$\begin{aligned}\sigma_{xy} - i\sigma_{yy} &= -\frac{2\sigma_0}{\sqrt{(1-\gamma^2)}}\frac{1}{\pi i}\int_{-1}^{1}\frac{w(t)P_1^{(\alpha,\beta)}(t)}{t-x}\mathrm{d}t\\ &= -2\sigma_0\sqrt{\left(\frac{1+\gamma}{1-\gamma}\right)}w(x)P_1^{(\alpha,\beta)}(x)+i\sigma_0\\ &= -\sigma_0(x-1)^\alpha(x+1)^\beta(2\omega+ix)+i\sigma_0; \qquad y = 0, \qquad x > 1.\end{aligned} \tag{29}$$

Also, defining the stress intensity factors by [1]

$$k_1+ik_2 = \lim_{x\to 1}(x-1)^{-\alpha}(x+1)^{-\beta}(\sigma_{yy}+i\sigma_{xy}) \tag{30}$$

we find

$$k_1 = \sigma_0, \qquad k_2 = -2\omega_0. \tag{31}$$

Similarly, in the general problem, for $|x| > 1$ substituting from (20) into (14) and taking into account (28), the interface stresses may be expressed as follows:

$$\begin{aligned}\frac{1+\kappa_3}{a_{21}\mu_3}(\sigma_{xy}-i\sigma_y) &= \frac{1}{\pi i}\int_{-1}^{1}\sum_1^\infty c_n w(t)P_n^{(\alpha,\beta)}(t)\frac{\mathrm{d}t}{t-x}\\ &+\int_{-1}^{1}\sum_1^\infty[K_1(x,t)c_n w(t)P_n^{(\alpha,\beta)}(t)\\ &+K_2(x,t)\bar{c}_n\overline{w(t)}\,\overline{P_n^{(\alpha,\beta)}(t)}]\,\mathrm{d}t, \qquad |x| > 1, \qquad y = 0.\end{aligned} \tag{32}$$

In the neighborhood of ∓ 1 the second integral of (32) is bounded, and the singular integrals may be evaluated as follows:

$$\frac{1}{\pi i}\int_{-1}^{1}\frac{w(t)P_n^{(\alpha,\beta)}(t)}{t-x}\mathrm{d}t = -(1+\gamma)[-w(x)P_n^{(\alpha,\beta)}(x)+G_n^{\infty}(x)], \qquad |x|>1 \tag{33}$$

where G_n^{∞} is the principal part of $wP_n^{(\alpha,\beta)}$ at infinity. Using (33) and noting that

$$w(x) = i\sqrt{\left(\frac{1-\gamma}{1+\gamma}\right)}(x-1)^{\alpha}(x+1)^{\beta}$$

from (32), (33) and (30) the stress intensity factors may be expressed as

$$k_1+ik_2 = -\frac{a_{21}\mu_3}{1+\kappa_3}\sqrt{(1-\gamma^2)}\sum_{1}^{\infty}c_nP_n^{(\alpha,\beta)}(1) \tag{34}$$

or, in terms of $\phi(x)$ we may write

$$k_1+ik_2 = -\lim_{x\to 1}\frac{a_{21}\mu_3}{1+\kappa_3}\sqrt{(1-\gamma^2)}\frac{\phi(x)}{w(x)} \tag{35}$$

which is the same result as that found in [1] and elsewhere. Note that if the crack is imbedded in a homogeneous medium, $\gamma = 0$, $\omega = 0$, $a_{21} = 2$, $P_n^{(\alpha,\beta)}(x) = P_n^{(-\frac{1}{2},-\frac{1}{2})}(x) = T_n(x)$, $w(x) = (1-x^2)^{-\frac{1}{2}}$ and (34) and (35) reduce to the expressions given in [1].

4. THE RESULTS

(a) As a first example we will consider the plane strain problem for two half-planes bonded through a layer which contains a crack on one of the interfaces ($h_2 = \infty$, Fig. 1).

TABLE 1. STRESS INTENSITY FACTORS FOR $h = \infty = h_2$ (FIGURE 1)

Material			$\frac{k_1}{\sigma_0\sqrt{a}}$	$\frac{k_2}{\sigma_0\sqrt{a}}$	$W_{2-3}(\infty) = \frac{k_1^2+k_2^2}{\sigma_0^2 a}$
2	3	ω			
Aluminum	Epoxy	0·0671	1	−0·1342	1·0180
Steel	Epoxy	0·07215	1	−0·1443	1·0208
Steel	Aluminum	0·04579	1	−0·09158	1·0084

TABLE 2. ELASTIC CONSTANTS

Material	E (lb in.$^{-2}$)	ν
Epoxy	4.5×10^5	0·35
Aluminum	10^7	0·3
Steel	3×10^7	0·3

For $p_1(x) = -\sigma_0, p_2(x) = 0$ and various material combinations considered in this paper the limiting values of the stress intensity factors obtained from (31) are given in Table 1. Table 2 shows the elastic constants used in the calculations.

For three different material combinations the stress intensity factors calculated from (34) as functions of relative layer thickness $h/2a$ are shown in Figs. 2–4. The figures also show the energy ratio, W related to the strain energy release rate $\partial U/\partial a$ as follows [9].

$$\left(\frac{\partial U}{\partial a}\right)_{2-3} = \frac{\pi}{2}\frac{1+\kappa_3}{a_{21}\mu_3}(k_1^2+k_2^2)$$

$$W_{2-3} = \frac{2\mu_3 a_{21}}{\sigma_0^2 a\pi(1+\kappa_3)}\left(\frac{\partial U}{\partial a}\right)_{2-3} = \frac{k_1^2+k_2^2}{\sigma_0^2 a}. \tag{36}$$

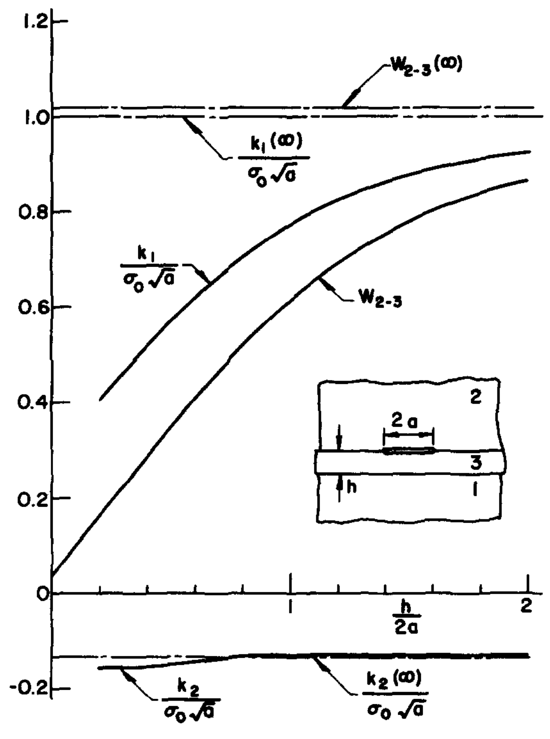

FIG. 2. Stress intensity factors and the strain energy release rate vs. $h/2a$. Materials: 1, steel; 2, aluminum; 3, epoxy.

Note that in the homogeneous case (i.e. $\mu_2 = \mu_3$, $\nu_2 = \nu_3$) $a_{21} = 2$ and the strain energy release rate is given by

$$\left(\frac{\partial U}{\partial a}\right)_3 = \frac{\pi(1+\kappa_3)}{4\mu_3}(k_1^2+k_2^2). \tag{37}$$

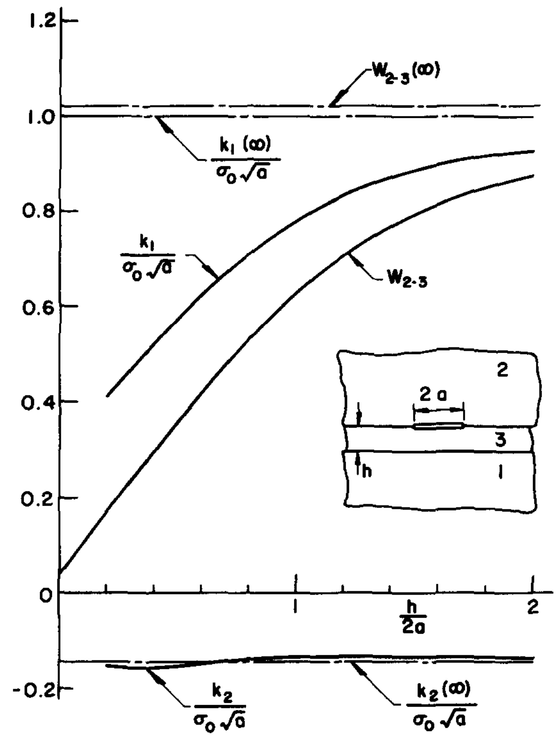

FIG. 3. Stress intensity factors and the strain energy release rate vs. $h/2a$. Materials: 1, aluminum; 2, steel; 3, epoxy.

For $h = 0$ the problem is that of two bonded half-planes containing an interface crack and the corresponding results are shown in Table 1. Thus, from

$$\lim_{h\to 0}\left(\frac{\partial U}{\partial a}\right)_{2-3} = \left(\frac{\partial U}{\partial a}\right)_{2-1}$$

we have

$$\lim_{h\to 0} \underset{2-3}{W(h)} = \frac{\mu_3}{\mu_1}\frac{1+\kappa_1}{1+\kappa_3}\frac{(a_{21})_{2-3}}{(a_{21})_{2-1}} W_{2-1}(\infty). \tag{38}$$

The other limiting value of W corresponding to $h = \infty$ is given in Table 1 and is also shown on the figures. These figures indicate that the strain energy release rate, which is the main load parameter used in the application of fracture theories, is very highly dependent on the layer-thickness to crack-length ratio, $h/2a$. If the modulus of the layer is smaller than that of the adjacent materials, $\partial U/\partial a$ decreases with decreasing $h/2a$. If the modulus of the layer is larger, this trend would be reversed.

Figure 5 shows some of the results of [1] combined with the limiting values obtained in this paper. In this problem the crack is in the layer, and the figure shows the variation of the energy ratio

$$W_3 = \frac{4\mu_3}{\sigma_0^2 a\pi(1+\kappa_3)}\left(\frac{\partial U}{\partial a}\right)_3 = \frac{k_1^2+k_2^2}{\sigma_0^2 a}$$

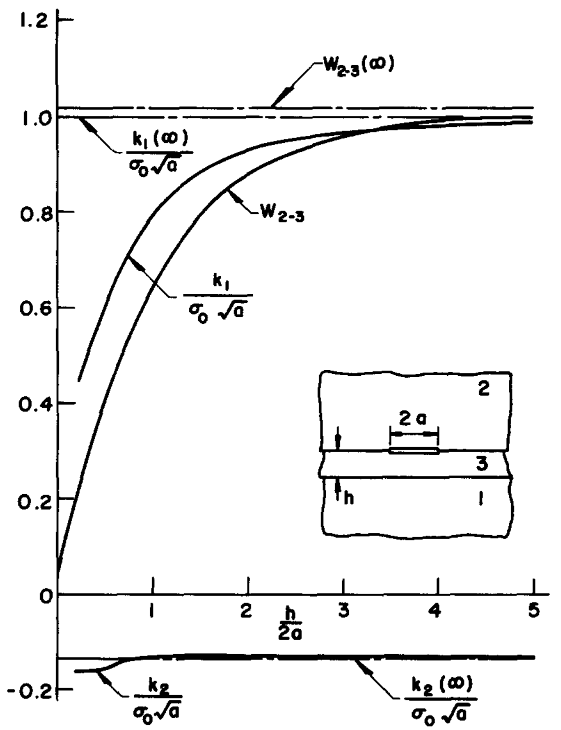

FIG. 4. Stress intensity factors and the strain energy release rate vs. $h/2a$. Materials: 1, aluminum; 2, aluminum; 3, epoxy.

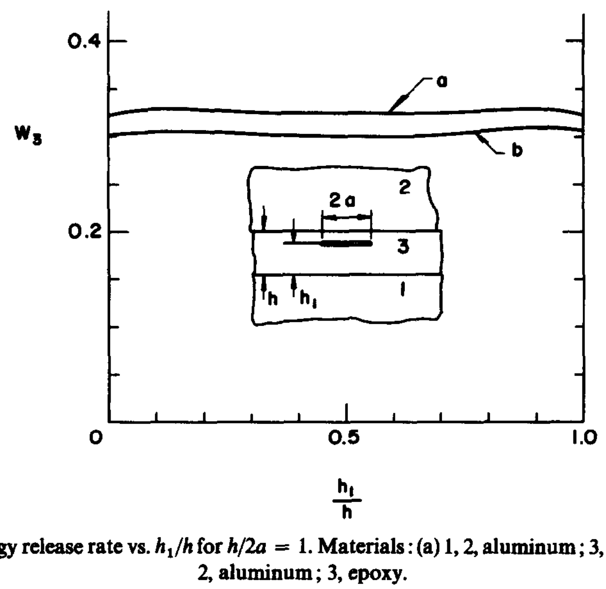

FIG. 5. Strain energy release rate vs. h_1/h for $h/2a = 1$. Materials: (a) 1, 2, aluminum; 3, epoxy; (b) 1, steel; 2, aluminum; 3, epoxy.

as a function of h_1/h for $h = 2a$, h_1 being the distance of the crack from the interface with material 1. The limiting values of W_3 shown in the figure are calculated from

$$\lim_{h_1\to h}\left(\frac{\partial U}{\partial a}\right)_3 = \left(\frac{\partial U}{\partial a}\right)_{2-3}, \qquad \lim_{h_1\to 0}\left(\frac{\partial U}{\partial a}\right)_3 = \left(\frac{\partial U}{\partial a}\right)_{1-3}$$

as follows:

$$\lim_{h_1\to h} W_3(h_1) = \frac{2}{(a_{21})_{2-3}} W_{2-3}(h)$$
$$\lim_{h_1\to 0} W_3(h_1) = \frac{2}{(a_{21})_{1-3}} W_{1-3}(h) \tag{39}$$

where, for the material combinations shown in Fig. 5, the corresponding values of W_{2-3} and W_{1-3} are given in Figs. 2–4 [at $(h/2a) = 1$].

Considering the fact that for the homogeneous infinite plane $W_3 = 1$, Fig. 5 shows the considerable decrease in the strain energy release rate resulting from the stiffer adjacent planes. However, unlike the anti-plane shear problem in which there is a rather sharp reduction in the strain energy release rate as the crack approaches the interfaces (see Fig. 3 of [1]), in the plane strain problem considered here the strain energy release rate remains relatively constant.

As mentioned in the previous section, the infinite system of algebraic equations for the constants c_n is solved by the method of reduction, i.e. the system is solved approximately by truncating the series (20) at the Nth term and considering only the first N equations of the system. To give an idea about the convergence of the computations, the stress intensity factors computed for various values of $h/2a$ and increasing values of N are shown in Table 3. The table corresponds to the load $\sigma_0 = 1$ and materials aluminum–epoxy–aluminum. It is clear from the table that the convergence is excellent. It should be noted that in problems of the type discussed here, the main computational effort goes into evaluating the Fredholm kernels, k_{ij} which are given in terms of infinite integrals such as (10). Unless these and the definite integrals which are used later to set up the algebraic system are calculated with sufficiently high accuracy, larger N in Table 3 would not produce more accurate results beyond certain number of significant digits.

From Figs. 2–4 we observe that as $h \to \infty$ the stress intensity factors k_1 and k_2 approach the asymptotic values given in Table 1 which are obtained in closed form. Obviously it would also be desirable to have such asymptotic values for k_1, k_2 as $h \to 0$. Analytically this would require the solution of the system of integral equations (11) in which the Fredholm kernels k_{ij} are replaced by the leading terms in their asymptotic expansion for small h (more specifically, the thickness-to-half crack length ratio). From the Appendix (1) it may easily be shown that for small h the leading terms of the functions $H_{ij}(\alpha)$ are

$$H_{ij}^0(\alpha) = b_{ij}\,e^{-2\alpha h}, \qquad (i,j = 1,2) \tag{40}$$

where the constants b_{ij} depend on μ_j, ν_j $(j = 1,2,3)$. Substituting from (40) into (10) the leading terms of the kernels k_{ij} are found to be

$$k_{11}^0(x,t) = b_{11}d(x,t,h), \qquad k_{12}^0(x,t) = b_{12}c(x,t,h)$$
$$k_{21}^0(x,t) = b_{21}c(x,t,h), \qquad k_{22}^0(x,t) = b_{22}d(x,t,h) \tag{41}$$
$$c(x,t,h) = \frac{t-x}{(2h)^2+(t-x)^2}, \qquad d(x,t,h) = \frac{2h}{(2h)^2+(t-x)^2}.$$

TABLE 3

N	4	6	8	10	12
$h/2a = 0{\cdot}2$					
k_1	0·4245	0·4357	0·4455	0·4449	0·4448
k_2	−0·1669	−0·1648	−0·1617	−0·1620	−0·1621
$h/2a = 0{\cdot}4$					
k_1	0·5576	0·5542	0·5524	0·5525	0·5525
k_2	−0·1611	−0·1603	−0·1598	−0·1598	−0·1598
$h/2a = 0{\cdot}6$					
k_1	0·6513	0·6497	0·6484	0·6484	0·6484
k_2	−0·1478	−0·1476	−0·1475	−0·1476	−0·1476

	$h/2a = 0{\cdot}8$			$h/2a = 1{\cdot}0$		
N	2	4	6	2	4	6
k_1	0·7356	0·7311	0·7300	0·7795	0·7931	0·7927
k_2	−0·1383	−0·1379	−0·1378	−0·1313	−0·1326	−0·1326

	$h/2a = 2{\cdot}0$			$h/2a = \infty$
N	2	4	6	1
k_1	0·9289	0·9302	0·9302	1
k_2	−0·1295	−0·1297	−0·1297	−0·1342

From (41) it is seen that as long as $h > 0$, no matter how small, k_{ij}^0 are bounded in the square domain $|t| \leq 1$, $|x| \leq 1$. Hence the dominant part and the fundamental function $w(x)$ of (11) remains to be that of materials 2 and 3. Given the present state of the theory of singular integral equations, with the Fredholm kernels k_{ij}^0, the closed form solution of (11) is not possible and the solution can only be obtained numerically.†

At $h = 0$ we have

$$\lim_{h\to 0^+} \int_{-1}^{1} f_j(t)c(x,t,h)\,\mathrm{d}t = \int_{-1}^{1} \frac{f_j(t)}{t-x}\,\mathrm{d}t$$
$$\lim_{h\to 0+} \int_{-1}^{1} f_j(t)d(x,t,h)\,\mathrm{d}t = \pi f_j(x) \tag{42}$$

which may be combined with the first two terms on the right hand side of (11). After some algebra it is easily shown that after this combination the resulting integral equations are the dominant system of two dissimilar bonded half planes 1 and 2. This system has a closed form solution [see (31) and Table 1]. Since the fundamental functions $w(x)$ for the two material

† Since for very small h the functions c and d given in (41) peak rather sharply around $t = x$, such a numerical scheme is bound to be unstable and may require great care. Further simplification of k_{ij}^0 in the form of narrow rectangular pulses would neither improve the stability nor make it possible to obtain a closed form solution.

combinations (1, 2) and (3, 2) are different, one should not expect a smooth transition between k_1, k_2 given in Table 1 and those obtained from (11) with the asymptotic kernels k_{ij}^0, and obviously there is no one set of asymptotic values of k_1 and k_2 for small values of h. However, it is also obvious that for any given small h k_1 and k_2 can be evaluated numerically to any desired degree of accuracy.† The quantity which must be a continuous function of h for $h \geq 0$ and which is physically more important is the strain energy release rate $\partial U/\partial a$. This is seen to be the case in all the examples worked out in this paper (see Figs. 2–5, 7).‡

(b) As a second example we consider an elastic layer bonded to a half-plane. The medium contains a crack on the interface and the external load is again assumed to be a uniform pressure $\sigma_{yy} = -\sigma_0$ on the crack surface (Fig. 6). In addition to its elastostatic structural applications, the solution may be useful as an approximation to the delamination problem caused by the reflected stress waves in layered materials. This is a special case of the previous problem in which μ_1 is taken to be zero. The results are shown in Fig. 6. It is seen that as the relative layer thickness approaches zero the stress intensity factors and the strain energy release rate go to infinity. The results for the other asymptotic case, i.e. $h/2a \to \infty$, shown in the figure are given in Table 1.

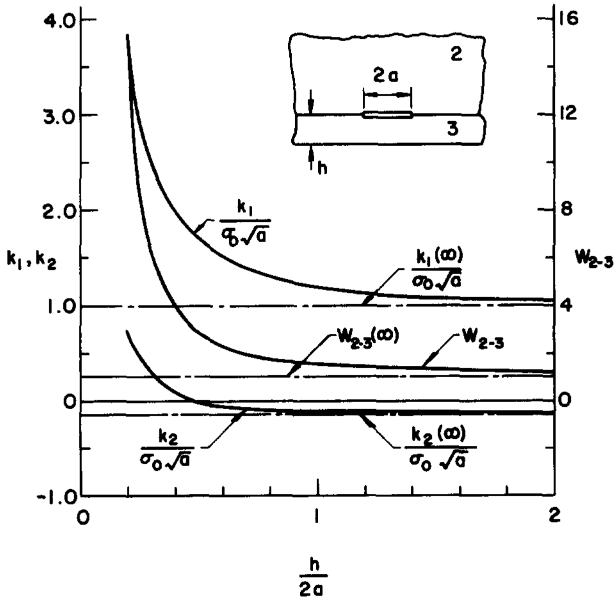

FIG. 6. Stress intensity factors and the strain energy release rate vs. $h/2a$. Materials: 2, aluminum; 3, epoxy.

† Even though Table 3 gives some indication of this, for $h \ll a$ it may be preferable to reduce (11) to a system of Fredholm equations.

‡ It should perhaps be mentioned that the case of a semi-infinite crack between the layer and one of the base materials may be considered as the limiting case of the problem for $h \ll a$. However, (a) physically this is an entirely different problem; the results obtained from its solution cannot readily be adapted to the problem posed in this paper, and (b) the closed form solution of it is not expected to be any simpler to obtain than that of the finite crack problem with asymptotic kernels k_{ij}^0; this is partly due to the fact that h is the only geometrical length in the medium and can be eliminated through normalization. One then has to solve the problem as is rather than trying to find an asymptotic solution for a "small parameter".

It should be noted that in all the examples discussed in this paper the stiffness of the material on the $y > 0$ side of the crack is greater than that of $y < 0$ side. Hence the interface shear around the crack tip $x = +1$ is always positive. On the other hand, in all cases the second component k_2 of the stress intensity factor is calculated to be negative. This should cause no confusion, since, unlike the homogeneous material, in the nonhomogeneous case the factors k_1 and k_2 are not directly identified with normal and shear stresses on the plane of the crack. This can be seen by expressing (30) around $x = 1$ as†

$$\sigma_{yy}+i\sigma_{xy}=\frac{k_1+ik_2}{\sqrt{(x^2-1)}}\left[\cos\left(\omega\log\frac{x+1}{x-1}\right)+i\sin\left(\omega\log\frac{x+1}{x-1}\right)\right]$$
$$+0(1), \qquad |x|>1, \qquad y=0 \tag{43}$$

where the results indicate that the dominant term for the shear stress is $k_1 \sin(\quad)$ rather than $k_2 \cos(\quad)$.

Figure 7 shows the strain energy release rate for the cracked layer obtained in [1] with the limiting value for $h_1 = h$ as obtained in this paper. The limit is evaluated in the same way as described for the previous example [see, (38) and (39)].

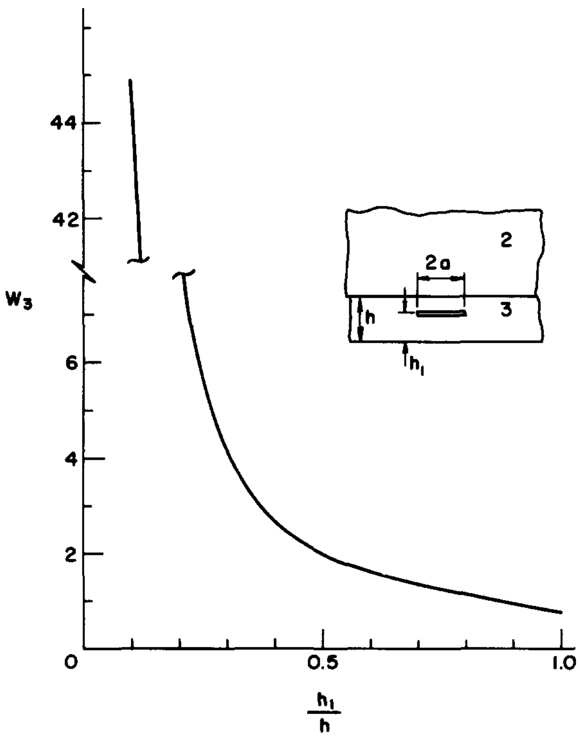

FIG. 7. Strain energy release rate vs. h_1/h for $h/2a = 1$. Materials: 2, aluminum; 3, epoxy.

† Note that ω as defined in (19) and β of [1] has opposite signs.

(c) Finally, as a last example we consider two bonded layers containing an interface crack which is opened by a uniform pressure $\sigma_{yy} = -\sigma_0$. The geometry, the relative dimensions, and the elastic constants as well as the results are shown in Fig. 8. The relative dimensions and the elastic constants used in this example roughly correspond to that of an

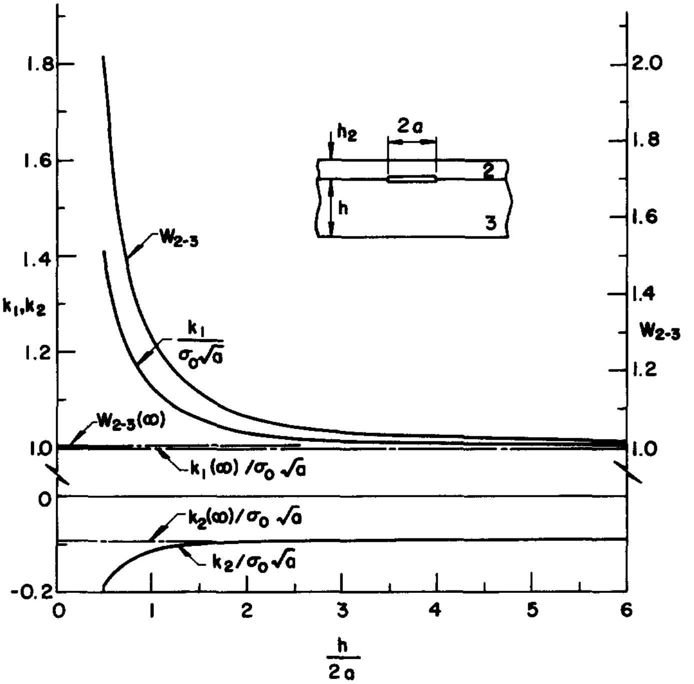

FIG. 8. Stress intensity factors and the strain energy release rate vs. $h_2/2a$ for $h/h_2 = 3$. Materials: 2, steel; 3, aluminum.

aluminum plate stiffened by a boron–epoxy composite layer as encountered in some applications. In this case too for increasing relative crack length or decreasing h_2/a ratio the stress intensity and the strain energy release rate ratios increase rather rapidly.

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APPENDIX. THE FUNCTIONS $H_{ij}(\alpha)$

(a) *Two half-planes bonded through a layer* (*Fig.* 1, $h_2 = \infty$)

$$H_{11}(\alpha) = \lambda_3 L_3 + (2\alpha h - 1)L_1 + e^{-2\alpha h}\left\{\lambda_1 L_1 + (1+2\alpha h)L_3 + \frac{1+2\alpha h}{\lambda_3\lambda_4}[2\alpha h(1+\lambda_4) - \lambda_4(1-\lambda_2)] - \frac{\lambda_1(1+\lambda_4)}{\lambda_4}\right\}$$

$$H_{12}(\alpha) = \lambda_3 L_4 + (2\alpha h - 1)L_2 + e^{-2\alpha h}\left\{\lambda_1 L_2 + (2\alpha h+1)L_4 - \frac{1+2\alpha h}{\lambda_3\lambda_4}[2\alpha h(1+\lambda_4) + \lambda_4(1-\lambda_2)] + \frac{\lambda_1(1+\lambda_4)}{\lambda_4}\right\}$$

$$H_{21}(\alpha) = -\lambda_3 L_3 - (2\alpha h + 1)L_1 + e^{-2\alpha h}\left\{\lambda_1 L_1 + (2\alpha h-1)L_3 + \frac{2\alpha h-1}{\lambda_3\lambda_4}[2\alpha h(1+\lambda_4) - \lambda_4(1-\lambda_2)] - \frac{\lambda_1(1+\lambda_4)}{\lambda_4}\right\}$$

$$H_{22}(\alpha) = -\lambda_3 L_4 - (2\alpha h + 1)L_2 + e^{-2\alpha h}\left\{\lambda_1 L_2 + (2\alpha h-1)L_4 - \frac{2\alpha h-1}{\lambda_3\lambda_4}[2\alpha h(1+\lambda_4) + \lambda_4(1-\lambda_2)] + \frac{\lambda_1(1+\lambda_4)}{\lambda_4}\right\}$$

$$L_1(\alpha) = \left(F_1 + \frac{1+\lambda_4}{\lambda_4}F_4\right)\Big/(\lambda_3\lambda_4 + F_4)$$

$$L_2(\alpha) = \left(F_2 - \frac{1+\lambda_4}{\lambda_4}F_4\right)\Big/(\lambda_3\lambda_4 + F_4)$$

$$L_3(\alpha) = \left(F_3 - \frac{2\alpha h(1+\lambda_4) - \lambda_4(1-\lambda_2)}{\lambda_3\lambda_4}F_4\right)\Big/(\lambda_3\lambda_4 + F_4)$$

$$L_4(\alpha) = \left(F_3 + \frac{2\alpha h(1+\lambda_4) + \lambda_4(1-\lambda_2)}{\lambda_3\lambda_4}F_4\right)\Big/(\lambda_3\lambda_4 + F_4)$$

$$F_1(\alpha) = e^{-2\alpha h}[2\alpha h(1-\lambda_2) - \lambda_2(1+\lambda_4)]$$

$$F_2(\alpha) = e^{-2\alpha h}[2\alpha h(1-\lambda_2) + \lambda_2(1+\lambda_4)]$$

$$F_3(\alpha) = -e^{-2\alpha h}(1-\lambda_2)\lambda_1$$

$$F_4(\alpha) = e^{-2\alpha h}(\lambda_1\lambda_3 + \lambda_2\lambda_4 - 4\alpha^2 h^2 + \lambda_1\lambda_2 e^{-2\alpha h})$$

$$\lambda_1 = (\kappa_1\mu_3 - \kappa_3\mu_1)/(\mu_1 + \kappa_1\mu_3)$$

$$\lambda_2 = (\kappa_2\mu_3 - \kappa_3\mu_2)/(\mu_2 + \kappa_2\mu_3)$$

$$\lambda_3 = (\mu_3 + \mu_1\kappa_3)/(\mu_1 - \mu_3)$$

$$\lambda_4 = (\mu_3 + \mu_2\kappa_3)/(\mu_2 - \mu_3).$$

(b) *Two bonded layers with an interface crack* (*Fig.* 1, $\mu_1 = 0$, *h and* h_2 *finite*)

$$H_{11}(\alpha) = \left\{2M_4 + \frac{2\alpha h_2\beta_1 - \beta_2}{\beta_1\beta_2}M_5 + (1-2\alpha h_2)\left(\frac{M_5}{\beta_2} - 2M_3\right)\right.$$

$$-2e^{-2\alpha h_2}[\beta_1 - M_3 + (2\alpha h_2+1)(\beta_1 2\alpha h_2 - \beta_2 + M_5)]\,(M_5 - 2\beta_1\beta_2)^{-1}$$

$$H_{12}(\alpha) = \left\{2M_2 + \frac{2\alpha h_2\beta_1 + \beta_2}{\beta_1\beta_2}M_5 + (1-2\alpha h_2)\left(\frac{M_5}{\beta_2} + 2M_1\right)\right.$$

$$\left. -2e^{-2\alpha h_2}[\beta_1 + M_1 + (2\alpha h_2+1)(2\alpha h_2\beta_1 + \beta_2 + M_2)]\right\}(M_5 - 2\beta_1\beta_2)^{-1}$$

$$H_{21}(\alpha) = \left\{2M_4 + \frac{2\alpha h_2\beta_1 - \beta_2}{\beta_1\beta_2}M_5 - (1+2\alpha h_2)\left(\frac{M_5}{\beta_2} - 2M_3\right)\right.$$

$$\left. +2e^{-2\alpha h_2}[\beta_1 - M_3 + (2\alpha h_2-1)(2\alpha h_2\beta_1 - \beta_2 + M_4)]\right\}(M_5 - 2\beta_1\beta_2)^{-1}$$

$$H_{22}(\alpha) = \left\{2M_2 + \frac{2\alpha h_2\beta_1 + \beta_2}{\beta_1\beta_2}M_5 - (1+2\alpha h_2)\left(\frac{M_5}{\beta_2} + 2M_1\right)\right.$$

$$\left. +2e^{-2\alpha h_2}[\beta_1 + M_1 + (2\alpha h_2-1)(2\alpha h_2\beta_1 - \beta_2 + M_4)]\right\}(M_5 - 2\beta_1\beta_2)^{-1}$$

$$\beta_1 = \frac{\mu_3 + \kappa_3\mu_2}{\mu_3},\ \beta_2 = \frac{\mu_2 + \kappa_2\mu_3}{\mu_2}$$

$$M_1(\alpha) = (\kappa_2 - 2\alpha h_2)\,e^{-2\alpha h_2} - \frac{\mu_2}{\mu_1(1+G_7)}[(2\alpha h - \kappa_3)G_5 - G_3$$

$$+\kappa_3 G_7 - e^{-2\alpha h} + G_1(2\alpha h + \kappa_3 - e^{-2\alpha h})]$$

$$M_2(\alpha) = e^{-2\alpha h_2} + \frac{\mu_2}{\mu_1(1+G_7)}[(2\alpha h - \kappa_3)G_6 - G_4 - (1+2\alpha h_2\kappa_3)G_7$$

$$+2\alpha h_2\,e^{-2\alpha h} + G_2\,(e^{-2\alpha h} + 2\alpha h + \kappa_3)]$$

$$M_3(\alpha) = -(2\alpha h_2 + \kappa_2)\,e^{-2\alpha h_2} + \frac{\mu_2}{\mu_1(1+G_7)}[(2\alpha h + \kappa_3)G_5 + G_3$$

$$+\kappa_3 G_7 - e^{-2\alpha h} + G_1(e^{-2\alpha h} - 2\alpha h + \kappa_3)]$$

$$M_4(\alpha) = -e^{-2\alpha h_2} + \frac{\mu_2}{\mu_1(1+G_7)}[(2\alpha h + \kappa_3)G_6 + G_4 + (1-2\alpha h_2\kappa_3)G_7$$

$$+2\alpha h_2\,e^{-2\alpha h} + G_2(e^{-2\alpha h} - 2\alpha h + \kappa_3)]$$

$$M_5(\alpha) = \beta_1[M_4 - M_2 + 2\alpha h_2(M_3 + M_1)] + \beta_2(M_3 - M_1) + M_2M_3 + M_1M_4$$

$$G_1(\alpha) = e^{-2\alpha h} + e^{-2\alpha h_2} + (4\alpha^2 hh_2 - 1)e^{-2\alpha(h_2+h)}$$

$$G_2(\alpha) = e^{-2\alpha h}(2\alpha h e^{-2\alpha h_2} - 2\alpha h_2 - 2\alpha h)$$

$$G_3(\alpha) = e^{-2\alpha h_2}(2\alpha h_2 + 2\alpha h - 2\alpha h_2\,e^{-2\alpha h})$$

$$G_4(\alpha) = e^{-2\alpha h} + e^{-2\alpha h_2} - e^{-2\alpha(h_1+h_2)}$$
$$G_5(\alpha) = e^{-2\alpha h}(G_3 - 2\alpha h)$$
$$G_6(\alpha) = e^{-2\alpha h}(G_4 - 1 + 4\alpha^2 h h_2)$$
$$G_7(\alpha) = -(2 + 4\alpha^2 h^2)e^{-2\alpha h} + e^{-4\alpha h}.$$

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Абстракт—Рассматривается плоская задача касающаяся деформации для соединенной среды, состоящей из трех разных материалов. Принимается, что среда обладает дефектом одной из поверхностей раздела, который можно рассматривать в смысле трещины. Получаются интегральные уравкения для общей задачи, которые являются системой сингулярных интегральных уравнений второго рода. Удаляется сингулярность системы. Решаются уравнения, учитывая факт, что фундаментальная Функция интегральных уравнений является функцией веса полиномов Якоби. В смысле специфичных примеров, решаются задачи для двух полуплоскостей, соединенных слоем, далее для упругого слоя, соединенного с полуплоскостью и для двух соединенных слоев. Даются численные результаты, указывающие факторы интенсивности напряжений и скорость выделения энергии деформации.